On representations of the exceptional superconformal algebra CK_6

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We realize the exceptional superconformal algebra CK_6, spanned by 32 fields, inside the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|3}$. We obtain a one-parameter family of irreducible representations of CK_6 in a superspace spanned by 8 fields.

1. Introduction

A *superconformal algebra* is a simple complex Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a finite family of pairwise local fields $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, one of which is the Virasoro field $L(z)$, [3, 8, 11]. Superconformal algebras play an important role in the string theory and conformal field theory.

The Lie superalgebras $K(N)$ of contact vector fields with Laurent polynomials as coefficients (with N odd variables) is a superconformal algebra which is characterized by its action on a contact 1-form, [3, 6, 8, 12]. These Lie superalgebras are also known to physicists as the $SO(N)$ superconformal algebras, [1]. Note that $K(N)$ is spanned by 2^N fields. It is simple if $N \neq 4$, if $N = 4$, then the derived Lie superalgebra $K'(4)$ is simple. The nontrivial central extensions of $K(1)$, $K(2)$ and $K'(4)$ are well-known: they are isomorphic to the so-called Neveu-Schwarz superalgebra, "the $N = 2$", and "the big $N = 4$" superconformal algebra, respectively, [1].

It was discovered independently in [3] and [17] that the Lie superalgebra of contact vector fields with polynomial coefficients in 1 even and 6 odd variables contains an exceptional simple Lie superalgebra (see also [6, 9, 10, 18, 19]).

In [3] a new exceptional superconformal algebra spanned by 32 fields was constructed as a subalgebra of $K(6)$, and it was denoted by CK_6. It was proven that CK_6 has no nontrivial central extensions. It was also pointed out that CK_6 appears to be the only new superconformal algebra, which completes their list (see [11, 12]).

In this work we realize CK_6 inside the Poisson superalgebra of pseudodifferential symbols on the supercircle $S^{1|3}$. It is known that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra, [2]. In particular, the Lie algebra $Vect(S^1)$ of complex polynomial vector fields on the circle has a natural embedding into the Poisson algebra P of formal Laurent series on the cylinder $T^*S^1 \setminus S^1$. One can consider a family of Lie algebras P_{h}, $\mathrm{h} \in]0,1]$, having the same underlying vector space, which contracts to P, [13-16].

Analogously, $K(2N)$ is embedded into the Poisson superalgebra $P(2N)$ of pseudodifferential symbols on the supercircle $S^{1|N}$, and there is a family of Lie superalgebras $P_{\mathrm{h}}(2N)$, which contracts to $P(2N)$ (see [20]).

A natural question is whether there exists an embedding

$$K(2N) \subset P_{\mathrm{h}}(2N). \tag{1.1}$$

Recall that the answer is "yes" if $N = 2$, more precisely, there exists an embedding of a nontrivial central extension of $K'(4) = [K(4), K(4)]$:

$$\hat{K}'(4) \subset P_{\mathrm{h}}(4). \tag{1.2}$$

Associated with this embedding, there is a one-parameter family of irreducible representations of $\hat{K}'(4)$ realized on 4 fields, [20].

Note that embedding (1.1) doesn't hold if $N > 2$, [5]. However, it is remarkable that it is possible to embed CK_6, which is "one half" of $K(6)$, into $P_{\mathrm{h}}(6)$. In this work we construct this embedding, and obtain the corresponding one-parameter family of representations of CK_6 realized on 8 fields.

2. Contact superconformal algebra $K(2N)$

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$, and let $\Lambda(1, 2N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$ be an associative superalgebra with natural multiplication and with the following parity of generators: $p(t) = \bar{0}$, $p(\xi_i) = p(\eta_i) = \bar{1}$ for $i = 1, \ldots, N$. Let $W(2N)$ be the Lie superalgebra of all derivations of $\Lambda(1, 2N)$. Let ∂_t, ∂_{ξ_i} and ∂_{η_i} stand for $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \xi_i}$ and $\frac{\partial}{\partial \eta_i}$, respectively. By definition,

$$K(2N) = \{D \in W(2N) \mid D\Omega = f\Omega \text{ for some } f \in \Lambda(1, 2N)\}, \tag{2.1}$$

where $\Omega = dt + \sum_{i=1}^N \xi_i d\eta_i + \eta_i d\xi_i$ is a differential 1-form, which is called a *contact form* (see [3, 4, 6, 7, 8, 10, 12]). The Euler operator is defined by $E = \sum_{i=1}^N \xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i}$.

We also define operators $\Delta = 2 - E$ and $H_f = (-1)^{p(f)+1} \sum_{i=1}^{N} \partial_{\xi_i} f \partial_{\eta_i} + \partial_{\eta_i} f \partial_{\xi_i}$, where $f \in \Lambda(1, 2N)$.

There is a one-to-one correspondence between the differential operators $D \in K(2N)$ and the functions $f \in \Lambda(1, 2N)$. The correspondence $f \leftrightarrow D_f$ is given by

$$D_f = \Delta(f)\frac{\partial}{\partial t} + \frac{\partial f}{\partial t} E - H_f. \tag{2.2}$$

The contact bracket on $\Lambda(1, 2N)$ is

$$\{f, g\}_K = \Delta(f)\partial_t g - \partial_t f \Delta(g) - \{f, g\}_{P.b}, \tag{2.3}$$

where

$$\{f, g\}_{P.b} = (-1)^{p(f)+1} \sum_{i=1}^{N} \partial_{\xi_i} f \partial_{\eta_i} g + \partial_{\eta_i} f \partial_{\xi_i} g \tag{2.4}$$

is the Poisson bracket. Thus $[D_f, D_g] = D_{\{f,g\}_K}$.

The superalgebra $K(6)$ contains an exceptional superconformal algebra, spanned by 32 fields, as a subalgebra. This superconformal algebra is denoted by CK_6 in [3, 8, 11]. Other notations are also used in the literature (see [6]). Let $\Theta = \xi_1\xi_2\xi_3\eta_1\eta_2\eta_3$. In what follows $(i, j, k) = (1, 2, 3)$ stays for the equality of cyclic permutations.

Proposition 1 (see [3, 6]). CK_6 is spanned by the following 32 fields:

$$\begin{aligned}
&L_n = t^{n+1} - (\partial_t)^3 t^{n+1}\Theta, \qquad (2.5)\\
&G_n^i = t^{n+1}\xi_i + (\partial_t)^2 t^{n+1}\partial_{\eta_i}\Theta, \quad \tilde{G}_n^i = t^n\eta_i + (\partial_t)^2 t^n \partial_{\xi_i}\Theta, \quad i = 1, 2, 3,\\
&T_n^{ij} = t^n\xi_i\eta_j - (\partial_t)t^n\partial_{\eta_i}\partial_{\xi_j}\Theta, \quad i \neq j, \quad T_n^i = t^n\xi_i\eta_i - (\partial_t)t^n\partial_{\eta_i}\partial_{\xi_i}\Theta, \quad i = 1, 2, 3,\\
&S_n^i = t^n\xi_i(\xi_j\eta_j + \xi_k\eta_k), \quad \tilde{S}_n^i = t^{n-1}\eta_i(\xi_j\eta_j - \xi_k\eta_k), \quad i = 1, 2, 3,\\
&I_n^i = t^{n-1}\xi_i\eta_j\eta_k, \quad i = 1, 2, 3, \quad I_n = t^{n+1}\xi_1\xi_2\xi_3,\\
&J_n^{ij} = t^{n+1}\xi_i\xi_j - (\partial_t)t^{n+1}\partial_{\eta_i}\partial_{\eta_j}\Theta, \quad \tilde{J}_n^{ij} = t^{n-1}\eta_i\eta_j - (\partial_t)t^{n-1}\partial_{\xi_i}\partial_{\xi_j}\Theta, \quad i < j,
\end{aligned}$$

where $n \in \mathbb{Z}$, and $(i, j, k) = (1, 2, 3)$ in the formulae for $S_n^i, \tilde{S}_n^i$ and I_n^i.

3. The Poisson superalgebra $P(2N)$ of pseudodifferential symbols on $S^{1|N}$

The *Poisson algebra P of pseudodifferential symbols on the circle* is formed by the formal series $A(t, \xi) = \sum_{-\infty}^{n} a_i(t)\xi^i$, where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable ξ corresponds to ∂_t. The Poisson bracket is defined as follows:

$$\{A(t, \xi), B(t, \xi)\} = \partial_\xi A(t, \xi)\partial_t B(t, \xi) - \partial_t A(t, \xi)\partial_\xi B(t, \xi). \tag{3.1}$$

The Poisson algebra P has a deformation P_{h}, where $\mathrm{h} \in]0,1]$. The associative multiplication in the vector space P is determined as follows:

$$A(t,\xi) \circ_{\mathrm{h}} B(t,\xi) = \sum_{n\geq 0} \frac{\mathrm{h}^n}{n!} \partial_\xi^n A(t,\xi) \partial_t^n B(t,\xi). \tag{3.2}$$

The Lie algebra structure on P_{h} is given by $[A,B]_{\mathrm{h}} = A \circ_{\mathrm{h}} B - B \circ_{\mathrm{h}} A$, so that the family P_{h} contracts to P. $P_{\mathrm{h}=1}$ is called the *Lie algebra of pseudodifferential symbols on the circle*, [13-16].

The *Poisson superalgebra* $P(2N)$ *of pseudodifferential symbols on the supercircle* $S^{1|N}$ has the underlying vector space $P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

$$\{A,B\} = \partial_\xi A \partial_t B - \partial_t A \partial_\xi B + \{A,B\}_{P.b}. \tag{3.3}$$

Let $\Lambda_{\mathrm{h}}(2N)$ be an associative superalgebra with generators $\xi_1,\dots,\xi_N,\eta_1,\dots,\eta_N$ and relations: $\xi_i\xi_j = -\xi_j\xi_i, \eta_i\eta_j = -\eta_j\eta_i, \eta_i\xi_j = \mathrm{h}\delta_{i,j} - \xi_j\eta_i$. Let $P_{\mathrm{h}}(2N) = P_{\mathrm{h}} \otimes \Lambda_{\mathrm{h}}(2N)$ be an associative superalgebra, where the product of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$, where $A_1, B_1 \in P_{\mathrm{h}}$, and $X, Y \in \Lambda_{\mathrm{h}}(2N)$, is given by

$$AB = \frac{1}{\mathrm{h}}(A_1 \circ_{\mathrm{h}} B_1) \otimes (XY). \tag{3.4}$$

Correspondingly, the Lie bracket in $P_{\mathrm{h}}(2N)$ is $[A,B]_{\mathrm{h}} = AB - (-1)^{p(A)p(B)}BA$, and $\lim_{\mathrm{h}\to 0}[A,B]_{\mathrm{h}} = \{A,B\}$. There exist natural embeddings: $W(N) \subset P(2N)$ and $W(N) \subset P_{\mathrm{h}}(2N)$, where $W(N)$ is the Lie superalgebra of all derivations of $\mathbb{C}[t,t^{-1}] \otimes \Lambda(\xi_1,\dots,\xi_N)$, so that the commutation relations in $P(2N)$ and in $P_{\mathrm{h}}(2N)$, when restricted to $W(N)$, coincide with the commutation relations in $W(N)$. $P_{\mathrm{h}=1}(2N)$ is called the *Lie superalgebra of pseudodifferential symbols on* $S^{1|N}$ (see [20]).

4. Realization of CK_6 inside the Poisson superalgebra

Theorem 1. The superalgebra CK_6 is spanned by the following 32 fields inside $P(2N)$:

$$\begin{aligned}
&L_{n,0} = t^{n+1}\xi, \qquad\qquad (4.1)\\
&G^i_{n,0} = t^{n+1}\xi\xi_i, \quad \tilde{G}^i_{n,0} = t^n\eta_i - nt^{n-1}\xi^{-1}\xi_j\eta_i\eta_j, \quad i = 1,2,3,\\
&T^{ij}_{n,0} = t^n\xi_i\eta_j - nt^{n-1}\xi^{-1}\xi_k\xi_i\eta_k\eta_j, \quad i \neq j \neq k,\\
&T^i_{n,0} = -t^n(\xi_j\eta_j + \xi_k\eta_k) + nt^{n-1}\xi^{-1}\xi_j\xi_k\eta_j\eta_k, \quad i = 1,2,3,\\
&S^i_{n,0} = -t^n\xi_i(\xi_j\eta_j + \xi_k\eta_k) + nt^{n-1}\xi^{-1}\xi_i\xi_j\xi_k\eta_j\eta_k, \quad i = 1,2,3,\\
&\tilde{S}^i_{n,0} = t^{n-1}\xi^{-1}(\xi_j\eta_j - \xi_k\eta_k)\eta_i, \quad i = 1,2,3,\\
&I^i_{n,0} = t^{n-1}\xi^{-1}\xi_i\eta_j\eta_k, \quad i = 1,2,3, \quad I_{n,0} = t^{n+1}\xi\xi_1\xi_2\xi_3,\\
&J^{ij}_{n,0} = t^{n+1}\xi\xi_i\xi_j, \quad \tilde{J}^{ij}_{n,0} = t^{n-1}\xi^{-1}\eta_i\eta_j, \quad i < j,
\end{aligned}$$

where $n \in \mathbb{Z}$, and $(i, j, k) = (1, 2, 3)$ in the formulae for $\tilde{G}^i_{n,0}$, $T^i_{n,0}$, $S^i_{n,0}$, $\tilde{S}^i_{n,0}$, and $I^i_{n,0}$.
Proof. Note that there exists an embedding

$$K(2N) \subset P(2N), \quad N \geq 0, \tag{4.2}$$

see [20]. Consider a $\mathbb{Z}$-grading of the associative superalgebra

$$P(2N) = \oplus_{i \in \mathbb{Z}} P_i(2N) \tag{4.3}$$

defined by

$$\begin{aligned} &\deg\, \xi = \deg\, \eta_i = 1, \text{ for } i = 1, \ldots, N, \\ &\deg\, t = \deg\, \xi_i = 0, \text{ for } i = 1, \ldots, N. \end{aligned} \tag{4.4}$$

With respect to the Poisson bracket,

$$\{P_i(2N), P_j(2N)\} \subset P_{i+j-1}(2N). \tag{4.5}$$

Thus $P_1(2N)$ is a subalgebra of $P(2N)$, and we will show that $P_1(2N) \cong K(2N)$. Equivalently, $P_1(2N)$ is singled out as the set of all (Hamiltonian) functions $A(t, \xi, \xi_i, \eta_i) \in P(2N)$ such that the corresponding vector fields supercommute with the semi-Euler operator:

$$[H_A, \xi\partial_\xi + \sum_{i=1}^{N} \eta_i \partial_{\eta_i}] = 0, \tag{4.6}$$

where

$$A(t, \xi, \xi_i, \eta_i) \longrightarrow H_A = \partial_\xi A \partial_t - \partial_t A \partial_\xi - (-1)^{p(A)} \sum_{i=1}^{N} (\partial_{\xi_i} A \partial_{\eta_i} + \partial_{\eta_i} A \partial_{\xi_i}). \tag{4.7}$$

To describe an isomorphism from $K(2N)$ onto $P_1(2N)$, we change the variable t in $\Lambda(1|2N)$: $t \xrightarrow{\chi} 2t - \sum_{i=1}^{N} \xi_i \eta_i$. Correspondingly, we have the following contact bracket on $\Lambda(1|2N)$:

$$\{f, g\}_{\tilde{K}} = \tilde{\Delta}(f)\partial_t g - \partial_t f \tilde{\Delta}(g) - \{f, g\}_{P.b}, \tag{4.8}$$

where $\tilde{\Delta} = 1 - \tilde{E}$ and $\tilde{E} = \sum_{i=1}^{N} \eta_i \partial_{\eta_i}$. Note that the corresponding contact form is $\tilde{\Omega} = dt + \sum_{i=1}^{N} \xi_i d\eta_i$. Define a map $\varphi : \Lambda(1|2N) \to P_1(2N)$ as follows:

$$f \xrightarrow{\varphi} A_f = (-1)^s \xi^{1-s} f, \tag{4.9}$$

where s is a scalar given by $\tilde{E}(f) = sf$. Then

$$\{A_f, A_g\} = A_{\{f,g\}_{\tilde{K}}}. \tag{4.10}$$

Applying the isomorphism $\psi = \varphi \circ \chi$ to the fields (2.5), we obtain the following fields:

$$\begin{aligned}
&\psi(L_n) = 2^{n+1} L_{n,0} - 2^{n-1}(n+1)(T^1_{n,0} + T^2_{n,0} + T^3_{n,0}), \quad \psi(G^i_n) = 2^{n+1} G^i_{n,0} - 2^n (n+1) S^i_{n,0},\\
&\psi(\tilde{G}^i_n) = -2^n \tilde{G}^i_{n,0} + 2^{n-1} n \tilde{S}^i_{n,0}, \quad \psi(T^{ij}_n) = -2^n T^{ij}_{n,0}, \quad \psi(T^i_n) = 2^{n-1}(-T^i_{n,0} + T^j_{n,0} + T^k_{n,0}),\\
&\psi(S^i_n) = 2^n S^i_{n,0}, \quad \psi(\tilde{S}^i_n) = 2^{n-1} \tilde{S}^i_{n,0}, \quad \psi(I^i_n) = 2^{n-1} I^i_{n,0}, \quad \psi(I_n) = 2^{n+1} I_{n,0},\\
&\psi(J^{ij}_n) = 2^{n+1} J^{ij}_{n,0}, \quad \psi(\tilde{J}^{ij}_n) = 2^{n-1} \tilde{J}^{ij}_{n,0}.
\end{aligned} \tag{4.11}$$

□

5. Realization of CK_6 inside the Lie superalgebra of pseudodifferential symbols

Given the embedding (4.2) it is natural to ask whether there exists an embedding

$$K(2N) \subset P_{\mathrm{h}}(2N). \tag{5.1}$$

Recall that if $N = 2$, then there is an embedding

$$\hat{K}'(4) \subset P_{\mathrm{h}}(4), \tag{5.2}$$

where $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \to K'(4) \to K(4) \to \mathbb{C} D_{t^{-1}\xi_1\xi_2\eta_1\eta_2} \to 0, \tag{5.3}$$

and $\hat{K}'(4)$ is a nontrivial central extension of $K'(4)$ (see [20]). The superalgebra $K'(4) \subset P(4)$ is spanned by the 12 fields:

$$f(\xi_1, \xi_2, t)\xi \text{ and } f(\xi_1, \xi_2, t)\eta_i \quad (i = 1, 2), \tag{5.4}$$

which form a subalgebra isomorphic to $W(2)$, together with 4 fields: F^i_n, where $i = 0, 1, 2, 3$, and $n \in \mathbb{Z}$:

$$\begin{aligned}
F^0_n &= t^{n-1}\xi^{-1}\eta_1\eta_2,\\
F^i_n &= t^{n-1}\xi^{-1}\xi_i\eta_1\eta_2, \quad i = 1, 2,\\
F^3_n &= t^{n-1}\xi^{-1}\xi_1\xi_2\eta_1\eta_2, \quad n \neq 0.
\end{aligned} \tag{5.5}$$

Proposition 2 ([20]). The superalgebra $\hat{K}'(4)$ in (5.2) is spanned by the 12 fields given in (5.4) together with 4 fields $F^i_{n,\mathrm{h}}$:

$$
\begin{aligned}
F^0_{n,\mathrm{h}} &= (\xi^{-1} \circ_{\mathrm{h}} t^{n-1})\eta_1\eta_2, \\
F^i_{n,\mathrm{h}} &= (\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_1\eta_2\xi_i, \quad i = 1, 2, \\
F^3_{n,\mathrm{h}} &= (\xi^{-1} \circ_{\mathrm{h}} t^{n-1})\eta_1\eta_2\xi_1\xi_2 + \frac{\mathrm{h}}{n} t^n, \quad n \neq 0,
\end{aligned}
\tag{5.6}
$$

and the central element h, so that $\lim_{\mathrm{h}\to 0} \hat{K}'(4) = K'(4) \subset P(4)$.

Note that we cannot obtain the embedding (5.1) if $N > 2$, [5]. However, the following theorem holds.

Theorem 2. There exists an embedding $CK_6 \subset P_{\mathrm{h}}(6)$ for each $\mathrm{h} \in]0, 1]$ such that $\lim_{\mathrm{h}\to 0} CK_6 = CK_6 \subset P(6)$.

Proof. CK_6 is spanned by the following fields inside $P_{\mathrm{h}}(6)$:

$$
\begin{aligned}
& L_{n,\mathrm{h}} = t^{n+1}\xi, \\
& G^i_{n,\mathrm{h}} = t^{n+1}\xi\xi_i, \quad \tilde{G}^i_{n,\mathrm{h}} = t^n\eta_i - n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_i\eta_j\xi_j, \quad i = 1, 2, 3, \\
& T^{ij}_{n,\mathrm{h}} = t^n\xi_i\eta_j - n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_k\eta_j\xi_k\xi_i, \quad i \neq j \neq k, \\
& T^i_{n,\mathrm{h}} = -t^n(\xi_j\eta_j + \xi_k\eta_k) + n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_j\xi_k + \mathrm{h}t^n, \quad i = 1, 2, 3, \\
& S^i_{n,\mathrm{h}} = -t^n\xi_i(\xi_j\eta_j + \xi_k\eta_k) + n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_i\xi_j\xi_k + \mathrm{h}t^n\xi_i, \quad i = 1, 2, 3, \\
& \tilde{S}^i_{n,\mathrm{h}} = \xi^{-1} \circ_{\mathrm{h}} t^{n-1}(\eta_j\eta_i\xi_j - \eta_k\eta_i\xi_k), \quad i = 1, 2, 3, \\
& I^i_{n,\mathrm{h}} = \xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_i, \quad i = 1, 2, 3, \quad I_{n,\mathrm{h}} = t^{n+1}\xi\xi_1\xi_2\xi_3, \\
& J^{ij}_{n,\mathrm{h}} = t^{n+1}\xi\xi_i\xi_j, \quad \tilde{J}^{ij}_{n,\mathrm{h}} = \xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_i\eta_j, \quad i < j,
\end{aligned}
\tag{5.7}
$$

where $n \in \mathbb{Z}$, and $(i, j, k) = (1, 2, 3)$ in the formulae for $\tilde{G}^i_{n,\mathrm{h}}$, $T^i_{n,\mathrm{h}}$, $S^i_{n,\mathrm{h}}$, $\tilde{S}^i_{n,\mathrm{h}}$ and $I^i_{n,\mathrm{h}}$. Let $\mathrm{h} \in [0, 1]$. Set $J^{ij}_{n,\mathrm{h}} = -J^{ji}_{n,\mathrm{h}}$ and $\tilde{J}^{ij}_{n,\mathrm{h}} = -\tilde{J}^{ji}_{n,\mathrm{h}}$ for $i > j$. Given $\mathrm{h} \in [0, 1]$, set

$$
L_n := L_{n,\mathrm{h}}, \quad \ldots, \quad \tilde{J}^{ij}_n := \tilde{J}^{ij}_{n,\mathrm{h}}.
\tag{5.8}
$$

Recall that if $\mathrm{h} = 0$, then (5.8) gives elements (4.1). The nonvanishing commutation relations between the elements (5.8) are as follows: let $i \neq j \neq k$, then

$$
\begin{aligned}
& [L_n, L_m] = (m - n)L_{n+m}, [L_n, G^i_m] = (m - n)G^i_{n+m}, [L_n, \tilde{G}^i_m] = m\tilde{G}^i_{n+m}, \\
& [L_n, T^{ij}_m] = mT^{ij}_{n+m}, [L_n, T^i_m] = mT^i_{n+m}, [L_n, S^i_m] = mS^i_{n+m}, [L_n, \tilde{S}^i_m] = (m + n)\tilde{S}^i_{n+m},
\end{aligned}
\tag{5.9}
$$

$$[L_n, I^i_m] = (m+n)I^i_{n+m}, [L_n, I_m] = (m-n)I_{n+m}, [L_n, J^{ij}_m] = (m-n)J^{ij}_{n+m},$$
$$[L_n, \tilde{J}^{ij}_m] = (m+n)\tilde{J}^{ij}_{n+m}, [G^i_n, G^j_m] = (m-n)J^{ij}_{n+m}, [G^i_n, \tilde{G}^j_m] = mT^{ij}_{n+m},$$
$$[G^i_n, T^{ji}_m] = -G^j_{n+m} + mS^j_{n+m}, [G^i_n, T^i_m] = mS^i_{n+m}, [G^i_n, T^j_m] = G^i_{n+m},$$
$$[G^i_n, S^j_m] = J^{ij}_{n+m}, [G^i_n, \tilde{S}^j_m] = T^{ij}_{n+m}, [\tilde{G}^i_n, \tilde{G}^j_m] = (m-n)\tilde{J}^{ij}_{n+m},$$
$$[\tilde{G}^i_n, S^i_m] = T^i_{n+m}, [\tilde{G}^i_n, S^j_m] = T^{ji}_{n+m}, [\tilde{G}^i_n, \tilde{S}^j_m] = -\tilde{J}^{ij}_{n+m}, [\tilde{G}^i_n, J^{ij}_m] = G^j_{n+m},$$
$$[T^{ij}_n, T^{ji}_m] = T^i_{n+m} - T^j_{n+m}, [T^{ij}_n, T^{jk}_m] = T^{ik}_{n+m}, [T^{ij}_n, T^{ki}_m] = -T^{kj}_{n+m}, [T^{ij}_n, T^i_m] = -T^{ij}_{n+m},$$
$$[T^{ij}_n, T^j_m] = T^{ij}_{n+m}, [T^{ij}_n, S^j_m] = S^i_{n+m}, [T^{ij}_n, \tilde{S}^i_m] = \tilde{S}^j_{n+m}, [T^{ij}_n, \tilde{S}^k_m] = -2I^i_{n+m},$$
$$[T^{ij}_n, I^j_m] = -\tilde{S}^k_{n+m}, [T^{ij}_n, J^{jk}_m] = J^{ik}_{n+m}, [T^{ij}_n, \tilde{J}^{ik}_m] = -\tilde{J}^{jk}_{n+m}, [T^j_n, S^i_m] = -S^i_{n+m},$$
$$[T^j_n, \tilde{S}^i_m] = \tilde{S}^i_{n+m}, [T^i_n, I^i_m] = 2I^i_{n+m}, [T^i_n, I_m] = -2I_{n+m},$$
$$[T^i_n, J^{ij}_m] = [T^j_n, J^{ij}_m] = -J^{ij}_{n+m}, [T^k_n, J^{ij}_m] = -2J^{ij}_{n+m}, [T^i_n, \tilde{J}^{ij}_m] = [T^j_n, \tilde{J}^{ij}_m] = \tilde{J}^{ij}_{n+m},$$
$$[T^k_n, \tilde{J}^{ij}_m] = 2\tilde{J}^{ij}_{n+m}, [J^{ij}_n, \tilde{J}^{ij}_m] = T^k_{n+m}, [J^{ij}_n, \tilde{J}^{ik}_m] = -T^{jk}_{n+m}, [J^{ij}_n, \tilde{J}^{jk}_m] = T^{ik}_{n+m}.$$

Let $(i, j, k) = (1, 2, 3)$, then

$$[G^i_n, \tilde{G}^i_m] = L_{n+m} - mT^k_{n+m}, [G^i_n, \tilde{S}^i_m] = T^j_{n+m} - T^k_{n+m}, [G^i_n, I^j_m] = T^{jk}_{n+m}, [G^i_n, I^k_m] = -T^{kj}_{n+m},$$
$$[G^i_n, J^{jk}_m] = (m-n)I_{n+m}, [G^i_n, \tilde{J}^{jk}_m] = (m+n)I^i_{n+m}, [G^i_n, \tilde{J}^{ij}_m] = \tilde{G}^j_{n+m} - (n+m)\tilde{S}^j_{n+m},$$
$$[G^i_n, \tilde{J}^{ik}_m] = \tilde{G}^k_{n+m}, [\tilde{G}^i_n, T^{ij}_m] = \tilde{G}^j_{n+m} - n\tilde{S}^j_{n+m}, [\tilde{G}^i_n, T^{ik}_m] = \tilde{G}^k_{n+m} - (n+m)\tilde{S}^k_{n+m},$$
$$[\tilde{G}^i_n, T^{jk}_m] = (m+n)I^j_{n+m}, [\tilde{G}^i_n, T^{kj}_m] = (n-m)I^k_{n+m}, [\tilde{G}^i_n, T^j_m] = -\tilde{G}^i_{n+m} + m\tilde{S}^i_{n+m},$$
$$[\tilde{G}^i_n, T^k_m] = -\tilde{G}^i_{n+m}, [\tilde{G}^i_n, I^i_m] = \tilde{J}^{jk}_{n+m}, [\tilde{G}^i_n, I_m] = J^{jk}_{n+m}, [S^i_n, J^{jk}_m] = -2I_{n+m}, [S^i_n, \tilde{J}^{ij}_m] = -\tilde{S}^j_{n+m},$$
$$[S^i_n, \tilde{J}^{ik}_m] = \tilde{S}^k_{n+m}, [S^i_n, \tilde{J}^{jk}_m] = 2I^i_{n+m}, [\tilde{S}^i_n, J^{ij}_m] = S^j_{n+m}, [\tilde{S}^i_n, J^{ik}_m] = -S^k_{n+m}, [I^i_n, J^{jk}_m] = -S^i_{n+m},$$
$$[I_n, \tilde{J}^{ij}_m] = S^k_{n+m}. \tag{5.10}$$

□

6. Representation of CK_6 associated with its embedding into $P_{\mathbf{h=1}}(6)$

Recall that the embedding (5.2) for h = 1 allows to define a one-parameter family of spinor-like representations of $K'(4)$ in the superspace spanned by 2 even and 2 odd fields, where the central element h acts by the identity operator, [20].

Theorem 3. There exists a one-parameter family of irreducible representations of CK_6, depending on parameter $\mu \in \mathbb{C}$, in a superspace spanned by 4 even fields and 4 odd fields.

Proof. Let $V^\mu = t^\mu\mathbb{C}[t, t^{-1}] \otimes \Lambda(3)$, where $\Lambda(3) = \Lambda(\xi_1, \xi_2, \xi_3)$ is the Grassmann algebra,

and $\mu \in \mathbb{R} \setminus \mathbb{Z}$. Let $\{v^i_m, \hat{v}^i_m\}$, where $m \in \mathbb{Z}$ and $i = 1, 2, 3, 4$, be the following basis in V^μ:

$$v^i_m = \frac{t^{m+\mu}}{m+\mu}\xi_i, \quad \hat{v}^i_m = t^{m+\mu}\xi_j\xi_k, \quad i = 1, 2, 3, \quad v^4_m = \frac{t^{m+\mu}}{m+\mu}, \quad \hat{v}^4_m = -t^{m+\mu}\xi_1\xi_2\xi_3, \tag{6.1}$$

where $(i, j, k) = (1, 2, 3)$ in the formulae for $\hat{v}^i_m$. We define a representation of CK_6 in V^μ according to the formulae (5.7), where $\mathrm{h} = 1$. Namely, ξ_i is the operator of multiplication in $\Lambda(3)$, η_i is identified with ∂_{ξ_i}, and ξ^{-1} is identified with the anti-derivative:

$$\xi^{-1} g(t) = \int g(t)dt, \quad g \in t^\mu \mathbb{C}[t, t^{-1}]. \tag{6.2}$$

Notice that the formula

$$\xi^{-1} \circ_{\mathrm{h}=1} f = \sum_{n=0}^{\infty} (-1)^n (\xi^n f) \xi^{-n-1}, \tag{6.3}$$

where $f \in \mathbb{C}[t, t^{-1}]$, when applied to a function $g \in t^\mu\mathbb{C}[t, t^{-1}]$, corresponds to the formula of integration by parts:

$$\int fgdt = f \int gdt - f' \int\int gdt^2 + f'' \int\int\int gdt^3 - \dots. \tag{6.4}$$

The superalgebra CK_6 acts on V^μ as follows (see (5.8) for notations):

$$\begin{aligned}
&L_n(v^i_m) = (m+n+\mu)v^i_{m+n}, \quad L_n(\hat{v}^i_m) = (m+\mu)\hat{v}^i_{m+n}, \\
&G^i_n(v^4_m) = (m+n+\mu)v^i_{m+n}, \quad G^i_n(\hat{v}^i_m) = -(m+\mu)\hat{v}^4_{m+n}, \\
&G^i_n(v^j_m) = \hat{v}^k_{m+n}, \quad G^i_n(v^k_m) = -\hat{v}^j_{m+n}, \quad \tilde{G}^i_n(v^i_m) = v^4_{m+n}, \quad \tilde{G}^i_n(\hat{v}^4_m) = -\hat{v}^i_{m+n}, \\
&\tilde{G}^i_n(\hat{v}^j_m) = -(m+\mu)v^k_{m+n}, \quad \tilde{G}^i_n(\hat{v}^k_m) = (m+n+\mu)v^j_{m+n}, \\
&T^{ij}_n(v^j_m) = v^i_{m+n}, \quad T^{ij}_n(\hat{v}^i_m) = -\hat{v}^j_{m+n}, \quad T^i_n(v^i_m) = v^i_{m+n}, \quad T^i_n(v^4_m) = v^4_{m+n}, \\
&T^i_n(\hat{v}^i_m) = -\hat{v}^i_{m+n} \quad T^i_n(\hat{v}^4_m) = -\hat{v}^4_{m+n}, \quad S^i_n(v^4_m) = v^i_{m+n}, \quad S^i_n(\hat{v}^i_m) = \hat{v}^4_{m+n}, \\
&\tilde{S}^i_n(\hat{v}^j_m) = v^k_{m+n}, \quad \tilde{S}^i_n(\hat{v}^k_m) = v^j_{m+n}, \quad I^i_n(\hat{v}^i_m) = -v^i_{m+n}, \quad I_n(v^4_m) = -\hat{v}^4_{m+n}, \\
&J^{ij}_n(v^4_m) = \hat{v}^k_{m+n}, \quad J^{ij}_n(v^k_m) = -\hat{v}^4_{m+n}, \quad \tilde{J}^{ij}_n(\hat{v}^k_m) = -v^4_{m+n}, \quad \tilde{J}^{ij}_n(\hat{v}^4_m) = v^k_{m+n},
\end{aligned} \tag{6.5}$$

where $(i, j, k) = (1, 2, 3)$ in the formulae for $\tilde{G}^i_n$, $\tilde{S}^i_n$, J^{ij}_n, and $\tilde{J}^{ij}_n$. Formulae (6.5) define a one-parameter family of representations of CK_6 in $V^\mu = < v^i_m, \hat{v}^i_m \mid i = 1, \dots, 4, \ m \in \mathbb{Z} >$.

$\square$

Remark 1. We have posed the condition $\mu \in \mathbb{R} \setminus \mathbb{Z}$ in the definition of V^μ. However, formulae (6.5) actually define a representation of CK_6 in a superspace spanned by $v_m^i, \hat{v}_m^i$ for an arbitrary $\mu \in \mathbb{C}$. (See also section 8).

7. The second family of representations of CK_6

Note that the embedding of infinite-dimensional Lie superalgebras

$$CK_6 \subset K(6), \tag{7.1}$$

considered in this work, is naturally related to the embedding of finite-dimensional Lie superalgebras

$$\hat{\mathcal{P}}(4) \subset P(0|6). \tag{7.2}$$

Recall that $P(0|6)$ is the Poisson superalgebra with 6 odd generators: $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$, and the Poisson bracket is given by (2.4). The simple Lie superalgebra $\mathcal{P}(n)$ is defined as follows. Let $\tilde{\mathcal{P}}(n)$ be the Lie superalgebra, which preserves the odd nondegenerate supersymmetric bilinear form antidiag $(1_n, 1_n)$ on the $(n|n)$-dimensional superspace. Thus

$$\tilde{\mathcal{P}}(n) = \{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in \mathfrak{gl}(n), B \text{ and } C \text{ are } n \times n - \text{ matrices}, B^t = B, C^t = -C \}. \tag{7.3}$$

$\mathcal{P}(n)$ is a subalgebra of $\tilde{\mathcal{P}}(n)$ such that $A \in \mathfrak{sl}(n)$, [7]. A. Sergeev has proved that $\mathcal{P}(n)$ has a nontrivial central extension if and only if $N = 4$, see [18]. Note that dim $\hat{\mathcal{P}}(4) = (16|16)$. It was pointed out in [6, 18] that $\hat{\mathcal{P}}(4)$ has a family spin$_\lambda$ of $(4|4)$-dimensional irreducible representations. In fact, there exist two such families: they correspond to two families of embeddings of $\hat{\mathcal{P}}(4)$ into $P(0|6)$.

For every $\lambda \neq 0$ we can realize $\hat{\mathcal{P}}(4)$ inside $P(0|6)$ as follows:

$$\hat{\mathcal{P}}(4) = < L, G^i, \tilde{G}^i, T^{ij}, T^i, S^i, \tilde{S}^i, I^i, I, J^{ij}, \tilde{J}^{ij} >, \tag{7.4}$$

where

$$\begin{aligned} & L = \lambda, G^i = \lambda\eta_i, \tilde{G}^i = \xi_i, T^{ij} = \eta_i\xi_j, T^i = -\eta_j\xi_j - \eta_k\xi_k, \\ & S^i = -\eta_i(\eta_j\xi_j + \eta_k\xi_k), \tilde{S}^i = \frac{1}{\lambda}(\eta_j\xi_j - \eta_k\xi_k)\xi_i, \\ & I^i = \frac{1}{\lambda}\eta_i\xi_j\xi_k, I = \lambda\eta_1\eta_2\eta_3, J^{ij} = \lambda\eta_i\eta_j, \tilde{J}^{ij} = \frac{1}{\lambda}\xi_i\xi_j, \end{aligned} \tag{7.5}$$

so that the central element is L. Correspondingly, there is an embedding of $\hat{\mathcal{P}}(4)$ into $P_{\mathrm{h}}(0|6)$ given by

$$L_{\mathrm{h}} = \lambda, G^i_{\mathrm{h}} = \lambda\eta_i, \tilde{G}^i_{\mathrm{h}} = \xi_i, T^{ij}_{\mathrm{h}} = \eta_i\xi_j, T^i_{\mathrm{h}} = -\eta_j\xi_j - \eta_k\xi_k + \mathrm{h}, \tag{7.6}$$
$$S^i_{\mathrm{h}} = -\eta_i(\eta_j\xi_j + \eta_k\xi_k) + \mathrm{h}\eta_i, \tilde{S}^i_{\mathrm{h}} = \frac{1}{\lambda}(\xi_j\xi_i\eta_j - \xi_k\xi_i\eta_k),$$
$$I^i_{\mathrm{h}} = \frac{1}{\lambda}\xi_j\xi_k\eta_i, I_{\mathrm{h}} = \lambda\eta_1\eta_2\eta_3, J^{ij}_{\mathrm{h}} = \lambda\eta_i\eta_j, \tilde{J}^{ij}_{\mathrm{h}} = \frac{1}{\lambda}\xi_i\xi_j,$$

and $\lim_{\mathrm{h}\to 0}\hat{\mathcal{P}}(4) = \hat{\mathcal{P}}(4) \subset P(0|6)$. The nonvanishing commutation relations between the elements (7.5) and between the elements (7.6) are as in (5.9)-(5.10), where the indexes $m = n = 0$.

Associated to this embedding (for $\mathrm{h} = 1$) there is a family spin^1_λ of representations of $\hat{\mathcal{P}}(4)$ in the superspace $\Lambda(\xi_1, \xi_2, \xi_3)$. We choose the basis

$$v^i = \xi_i,\ \hat{v}^i = \frac{1}{\lambda}\xi_j\xi_k,\ i = 1, 2, 3,\ v^4 = 1,\ \hat{v}^4 = -\frac{1}{\lambda}\xi_1\xi_2\xi_3. \tag{7.7}$$

Explicitely,

$$\mathrm{spin}^1_\lambda : \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} + \mathbb{C}L \to \begin{pmatrix} A & B - \lambda\tilde{C} \\ C & -A^t \end{pmatrix} + \mathbb{C}\lambda \cdot 1_{4|4}, \tag{7.8}$$

where $1_{4|4}$ is the identity matrix, and if $C_{ij} = E_{ij} - E_{ji}$, then $\tilde{C}_{ij} = C_{kl}$, so that the permutation $(1, 2, 3, 4) \mapsto (i, j, k, l)$ is even, cf. [6, 18]. Formula (7.8) also gives the standard representation spin^1_0.

The second family of embeddings of $\hat{\mathcal{P}}(4)$ into $P(0|6)$ and into $P_{\mathrm{h}}(0|6)$ is given by (7.4)-(7.6), where ξ_i is interchanged with η_i for all i in all the formulae. Correspondingly, there is a family spin^2_λ of representations of $\hat{\mathcal{P}}(4)$ associated to this embedding (for $\mathrm{h} = 1$) in the superspace $\Lambda(\xi_1, \xi_2, \xi_3)$, so that $\Pi(\mathrm{spin}^2_\lambda) \cong \mathrm{spin}^1_\lambda$, as $\hat{\mathcal{P}}(4)$-modules, for all λ.
(Π denotes the change of parity). From Theorem 3 we have the following corollary.

Corollary 1. Under the restriction of the representation of CK_6 in V^μ to $\hat{\mathcal{P}}(4)$, V^μ decomposes into a direct sum of irreducible $(4|4)$-dimensional representations of the family spin^2_λ.

Proof. Naturally, there are embeddings:

$$\hat{\mathcal{P}}(4) \subset CK_6, \quad P(0|6) \subset K(6). \tag{7.9}$$

The first embedding is given as follows:

$$\hat{\mathcal{P}}(4) = \{x \in CK_6 \mid [L_0, x] = 0\}, \tag{7.10}$$

hence L_0 is the central element. The nontrivial 2-cocycle on $\mathcal{P}(4)$ is $(G_0^i, \tilde{G}_0^j) = \delta_{i,j} L_0$. It follows from (6.5) that V^μ is a direct sum of (4|4)-dimensional $\hat{\mathcal{P}}(4)$-submodules:

$$V^\mu = \oplus_{m\in\mathbb{Z}} V_m^\mu, \quad V_m^\mu = < v_m^i, \hat{v}_m^i \mid i = 1, 2, 3, 4 >, \tag{7.11}$$

where $V_m^\mu \cong \mathrm{spin}^2_{m+\mu}$.

$\square$

It is possible to define another embedding of CK_6 into $P(6)$ (respectively, into $P_{\mathrm{h}}(6)$) by interchanging ξ_i with η_i in all the formulae (4.1) (respectively in (5.7)), and then obtain a one-parameter family of representations of CK_6 in V^μ by repeating the previous construction. Thus the following theorem holds.

Theorem 4. Consider the following basis in V^μ:

$$v_m^i = t^{m+\mu}\xi_i, \quad \hat{v}_m^i = \frac{t^{m+\mu}}{m+\mu}\xi_j\xi_k, \quad i = 1, 2, 3, \quad v_m^4 = t^{m+\mu}, \quad \hat{v}_m^4 = -\frac{t^{m+\mu}}{m+\mu}\xi_1\xi_2\xi_3, \tag{7.12}$$

where $(i, j, k) = (1, 2, 3)$ in the formulae for $\hat{v}_m^i$. Then the action of CK_6 on V^μ is defined as follows

$$\begin{aligned}
&L_n(v_m^i) = (m+\mu)v_{m+n}^i, \quad L_n(\hat{v}_m^i) = (m+n+\mu)\hat{v}_{m+n}^i, \\
&G_n^i(v_m^i) = (m+\mu)v_{m+n}^4, \quad G_n^i(\hat{v}_m^4) = -(m+n+\mu)\hat{v}_{m+n}^i, \\
&G_n^i(\hat{v}_m^k) = v_{m+n}^j, \quad G_n^i(\hat{v}_m^j) = -v_{m+n}^k, \quad \tilde{G}_n^i(v_m^4) = v_{m+n}^i, \quad \tilde{G}_n^i(\hat{v}_m^i) = -\hat{v}_{m+n}^4, \\
&\tilde{G}_n^i(v_m^k) = -(m+n+\mu)\hat{v}_{m+n}^j, \quad \tilde{G}_n^i(v_m^j) = (m+\mu)\hat{v}_{m+n}^k, \\
&T_n^{ij}(v_m^i) = -v_{m+n}^j, \quad T_n^{ij}(\hat{v}_m^j) = \hat{v}_{m+n}^i, \quad T_n^i(v_m^i) = -v_{m+n}^i, \quad T_n^i(v_m^4) = -v_{m+n}^4, \\
&T_n^i(\hat{v}_m^i) = \hat{v}_{m+n}^i \quad T_n^i(\hat{v}_m^4) = \hat{v}_{m+n}^4, \quad S_n^i(v_m^i) = -v_{m+n}^4, \quad S_n^i(\hat{v}_m^4) = -\hat{v}_{m+n}^i, \\
&\tilde{S}_n^i(v_m^k) = -\hat{v}_{m+n}^j, \quad \tilde{S}_n^i(v_m^j) = -\hat{v}_{m+n}^k, \quad I_n^i(v_m^i) = \hat{v}_{m+n}^i, \quad I_n(\hat{v}_m^4) = v_{m+n}^4, \\
&J_n^{ij}(\hat{v}_m^k) = -v_{m+n}^4, \quad J_n^{ij}(\hat{v}_m^4) = v_{m+n}^k, \quad \tilde{J}_n^{ij}(v_m^4) = \hat{v}_{m+n}^k, \quad \tilde{J}_n^{ij}(v_m^k) = -\hat{v}_{m+n}^4,
\end{aligned} \tag{7.13}$$

where $(i, j, k) = (1, 2, 3)$ in the formulae for $\tilde{G}_n^i$, $\tilde{S}_n^i$, J_n^{ij}, and $\tilde{J}_n^{ij}$. Thus V^μ is a direct sum of (4|4)-dimensional $\hat{\mathcal{P}}(4)$-submodules, see (7.11), where $V_m^\mu \cong \mathrm{spin}^1_{m+\mu}$.

8. Final remarks

In Theorem 1 we realized CK_6 inside the deg $= 1$ part of the $\mathbb{Z}$-grading of $P(6)$, given by (4.3), and in Theorem 2 we realized CK_6 inside $P_{\mathrm{h}}(6)$. One should note that in this realization the elements of CK_6 have powers $-1, 0$ and 1 with respect to ξ, see (4.1) and (5.7).

We will now show how to single out CK_6 from $P_{\mathrm{h}}(6)$. Let S be a subspace of $P_{\mathrm{h}}(6)$ spanned by $W(3)$ (which consists of the elements of power 0 and 1 with respect to ξ) and the following fields ($n \in \mathbb{Z}$):

$$\begin{aligned}
&\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_i\eta_j, \quad \xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_i,\\
&\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_i\eta_j\xi_j, \quad \xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_k\eta_j\xi_k\xi_i,\\
&n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_j\xi_k + \mathrm{h}t^n, \quad n\xi^{-1} \circ_{\mathrm{h}} t^{n-1}\eta_j\eta_k\xi_i\xi_j\xi_k + \mathrm{h}t^n\xi_i.
\end{aligned} \tag{8.1}$$

Fix $\mathrm{h} = 1$. Let $\mu \in (0, 1)$. The action of the elements of S on the spaces V^μ is well-defined. In each V^μ we defined a basis by (6.1). We will denote it now by $V^\mu =< v^i_m(\mu), \hat{v}^i_m(\mu) >$. Let $v(\mu) \in V^\mu$ be vectors which have the same coordinates with respect to this basis for all μ. Consider an odd nondegenerate superskew-symmetric form on each V^μ:

$$\begin{aligned}
&(v^i_m(\mu), \hat{v}^i_l(\mu))_\mu = -(\hat{v}^i_l(\mu), v^i_m(\mu))_\mu = \delta_{m+l,0} \quad i = 1, 2, 3.\\
&(v^4_m(\mu), \hat{v}^4_l(\mu))_\mu = -(\hat{v}^4_l(\mu), v^4_m(\mu))_\mu = -\delta_{m+l,0}.
\end{aligned} \tag{8.2}$$

Let $V =< v^i_m, \hat{v}^i_m >$, where $i = 1, \dots, 4$, $m \in \mathbb{Z}$, be a superspace such that $p(v^i_m) = p(\hat{v}^4_m) = \bar{1}, p(\hat{v}^i_m) = p(v^4_m) = \bar{0}$. A superskew-symmetric form on V is defined by

$$\begin{aligned}
&(v^i_m, \hat{v}^i_l) = -(\hat{v}^i_l, v^i_m) = \delta_{m+l,0} \quad i = 1, 2, 3,\\
&(v^4_m, \hat{v}^4_l) = -(\hat{v}^4_l, v^4_m) = -\delta_{m+l,0}.
\end{aligned} \tag{8.3}$$

Theorem 5.

$$\begin{aligned}
&CK_6 = \{X \in S \mid \lim_{\mu\to 0}[(Xv(\mu), w(\mu))_\mu + (-1)^{p(X)p(v(\mu))}(v(\mu), Xw(\mu))_\mu] = 0,\\
&\text{for all } v(\mu), w(\mu) \in V^\mu\}.
\end{aligned} \tag{8.4}$$

There is a representation of CK_6 in V given by (6.5), where $\mu = 0$, and this action preserves the form (8.3).

Remark 2. Correspondingly, there is a representation of CK_6 in V given by (7.13), where $\mu = 0$, and this action preserves the odd nondegenerate supersymmetric form on V:

$$(v^i_m, \hat{v}^i_l) = (\hat{v}^i_l, v^i_m) = \delta_{m+l,0} \quad i = 1, 2, 3, 4. \tag{8.5}$$

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